Cannon for Neutral Particles

V. I. Yukalov and E. P. Yukalova
Bogolubov Laboratory of Theoretical Physics
Joint Institute for Nuclear Research, Dubna 141980, Russia

Dynamics of spin–polarized neutral particles, such as neutrons or neutral atoms and molecules, in magnetic fields is studied. A new regime of motion is found where particles move mainly in one direction forming a well–collimated beam. This regime suggests a mechanism for creating devices emitting directed beams of neutral particles.

Keywords: dynamics of neutral particles, nonadiabatic initial conditions

PACS: 03.75.–b, 07.77.Gx, 39.10.+j, 41.85.–p

The procedure of creating and accelerating directed beams of particles is of high practical importance. It would be difficult to enumerate various applications of such directed beams. Charged particles, as is known, are well manipulated by electric and magnetic fields [1]. To the contrary, for neutral particles there are no known ways of creating and accelerating directed beams by means of nonresonant electromagnetic fields. Directed neutral beams have been formed by employing mechanical collimators selecting particles from an isotropic distribution, by accelerating neutral particles through long tubes with a high pressure difference between the ends, as is done in molecular beam masers [2-4], and for resonance atoms by using laser beams. In this paper we show that there exists a *magnetic* mechanism that allows the formation of *directed* beams of neutral particles with strong acceleration. This mechanism requires no mechanical collimators, but does require a particular configuration of magnetic fields and a special polarization of particle spins at the initial time. A brief announcement of the results presented below has been given in Ref. [5].

To demonstrate the principal possibility of this new dynamical mechanism, we consider a rarefied gas of particles whose collisions at the first approximation can be neglected. Let particles be placed in magnetic fields whose space variation is sufficiently smooth, so that the semiclassical description can be applied [6]. Then for a particle with mass m and magnetic moment μ_0, one may write the evolution equations for quantum–mechanical averages. For the average real–space variable $\vec{R}=\{R^x, R^y, R^z\}$ one has

$$\frac{d^2 R^\alpha}{dt^2} = \frac{\mu_0}{m}\,\vec{S}\cdot\frac{\partial\vec{B}}{\partial R^\alpha}\,, \qquad (\alpha = x, y, z) \tag{1}$$

with initial conditions $\vec{R}(0)=\vec{R}_0$ and $\dot{\vec{R}}(0)=\dot{\vec{R}}_0$, where the dot means, as usual, a time derivative. It is easy to check [6] that Eq. (1) follows from the definition of $\vec{R}$ as an average of a position operator with a wave function satisfying the time–dependent Schrödinger equation. The average spin $\vec{S}=\{S^x, S^y, S^z\}$ satisfies the equation

$$\frac{d\vec{S}}{dt} = \frac{\mu_0}{\hbar}\vec{S}\times\vec{B}, \tag{2}$$

with an initial condition $\vec{S}(0) = \{S_0^x, S_0^y, S_0^z\}$. Let us take the total magnetic field $\vec{B}=\vec{B}_1 + \vec{B}_2$ as a sum of two terms,

$$\vec{B}_1 = B_1'\left(R^x\vec{e}_x + R^y\vec{e}_y + \lambda R^z\vec{e}_z\right), \qquad \vec{B}_2 = B_2\vec{h}(t), \tag{3}$$

the first being a quadrupole field, parametrized by its gradient B_1' and the anisotropy parameter λ, and the second, a transverse field, with

$$\vec{h}(t) = h_x\vec{e}_x + h_y\vec{e}_y, \qquad |\vec{h}| = 1, \tag{4}$$

depending only on time but not on real–space variables. Such magnetic fields are easy to form and are often used in different applications. For example, the quadrupole fields are the basis of quadrupole magnetic traps and the Ioffe–Pritchard traps [7,8]. A transverse *rotating* field, with $h_x = \cos\omega t$ and $h_y = \sin\omega t$, has also been employed in magnetic traps [9]. This rotating field will be used below as a concrete example. However, the effect we consider exists not solely for such a rotating field but for a wide class of transverse fields satisfying the condition

$$\left|d\vec{h}/dt\right| \ll \mu_0 B_2/\hbar\ . \tag{5}$$

To be absolutely concrete, we also take the anisotropy parameter $\lambda = -2$ so that $\vec{\nabla}\cdot\vec{B}_1 = 0$. For what follows, it is convenient to measure the components of the space vector $\vec{R}$ in units of the characteristic length $R_0 \equiv B_2/B_1'$ defining the radius of the field zero in the radial direction. To this end, we define the dimensionless vector $\vec{r}\equiv\vec{R}/R_0 = \{x,y,z\}$. Introduce the characteristic frequencies

$$\omega_1^2 \equiv \mu_0 B_1'/mR_0, \qquad \omega_2 \equiv \mu_0 B_2/\hbar, \tag{6}$$

the first of which, ω_1, corresponds to the motion of particles in real space and the second, ω_2, to the spin motion. The physical meaning of these frequencies becomes evident after we present Eqs. (1) and (2) in the form

$$\frac{d^2\vec{r}}{dt^2} = \omega_1^2\left(S^x\vec{e}_x + S^y\vec{e}_y - 2S^z\vec{e}_z\right), \qquad \frac{d\vec{S}}{dt} = \omega_2\hat{A}\vec{S}, \tag{7}$$

where the matrix $\hat{A} = [A_{\alpha\beta}]$, with $\alpha,\beta = 1,2,3$, consists of the elements $A_{\alpha\alpha} = 0$, $A_{12} = -A_{21} = -2z$, $A_{13} = -A_{31} = -y - h_y$, $A_{23} = -A_{32} = x + h_x$. From (7) it is really clear that ω_1 and ω_2 are the characteristic frequencies of the space and spin motions, respectively.

Notice that the system of equations (7), with a nonuniform magnetic field, is invariant under the change $\vec{S}\rightarrow -\vec{S}$ and $\vec{r}\rightarrow -\vec{r}$. This invariance can be called the Stern–Gerlach symmetry since in the particular case of a uniform magnetic field one would recover the conditions of the Stern–Gerlach experiment.

To solve the system of nonlinear differential equations (7), let us recall that (1) and (2) are derived in the semiclassical approximation whose criterion of validity is the slow space variation of magnetic fields [6]. In our notation, this criterion can be written as the inequality $|\omega_1/\omega_2| \ll 1$. This condition shows that the space variable $\vec{r}$ can be treated as slow, compared to the fast spin variable $\vec{S}$. From inequality (5) it follows that $\vec{h}$ is also slow as compared to $\vec{S}$. For the rotating field, condition (5) simply means that $\omega \ll \omega_2$. Therefore, the system of nonlinear equations (7) can be solved by employing the method of scale separation [10-13] which is a variant of the Krylov–Bogolubov averaging method [14,15]. A detailed description of this approach as applied to the nonadiabatic dynamics of atoms in nonuniform magnetic fields has been given in Refs. [16,17]. Following the method of scale separation, we, first, need to solve the equation for the fast variable, keeping there the slow variables as quasi–integrals of the motion. Then, under fixed $\vec{r}$ and $\vec{h}$, the second equation from (7) can be solved exactly. The resulting solution is

$$\vec{S}(t) = \sum_{i=1}^{3} a_i\vec{S}_i(t), \qquad a_i = \vec{S}(0)\cdot\vec{b}_i(0), \qquad \vec{S}_i(t) = \vec{b}_i(t)\exp\{\beta_i(t)\}, \tag{8}$$

$$\vec{b}_i(t) = \frac{1}{\sqrt{C_i}}\left[(\alpha_i A_{13} + A_{12}A_{23})\vec{e}_x + (\alpha_i A_{23} - A_{12}A_{13})\vec{e}_y + \left(\alpha_i^2 + A_{12}^2\right)\vec{e}_z\right],$$

$$C_i = \left(|\alpha_i|^2 - A_{12}^2\right)^2 + \left(|\alpha_i|^2 + A_{12}^2\right)\left(A_{13}^2 + A_{23}^2\right), \qquad \alpha^2 = A_{12}^2 + A_{13}^2 + A_{23}^2 ,$$

$$\alpha_{1,2} = \pm i\alpha, \qquad \alpha_3 = 0, \qquad \beta_i(t) = \omega_2 \int_0^t \alpha_i(t)\, dt , \qquad \beta_3(t) = 0.$$

The fast solution (8) is to be substituted into the equation for the slow variable, averaging the right–hand side of it over an interval of time much longer than the period of fast oscillations $2\pi/\omega_2$, which gives $\langle\vec{S}\rangle = a_3\langle\vec{b}_3\rangle$, and for the case of the rotating field

$$a_3 = \frac{(1+x)S_0^x + yS_0^y - 2zS_0^z}{[(1+x)^2 + y^2 + 4z^2]^{1/2}}, \qquad \vec{b}_3 = \frac{(x+\cos\omega t)\,\vec{e}_x + (y+\sin\omega t)\,\vec{e}_y - 2z\,\vec{e}_z}{[1 + 2(x\cos\omega t + y\sin\omega t) + x^2 + y^2 + 4z^2]^{1/2}}. \tag{9}$$

In this way, we obtain

$$\langle\omega_1^2\left(S^x\,\vec{e}_x + S^y\,\vec{e}_y - 2S^z\,\vec{e}_z\right)\rangle = \frac{\omega_1^2\,[(1+x)S_0^x + yS_0^y - 2zS_0^z]\left(x\,\vec{e}_x + y\,\vec{e}_y + 8z\,\vec{e}_z\right)}{2\left[(1+2x+x^2+y^2+4z^2)(1+x^2+y^2+4z^2)\right]^{1/2}}. \tag{10}$$

As a result, we come to the equation

$$\frac{d^2\vec{r}}{dt^2} = <\omega_1^2(S^x\,\vec{e}_x + S^y\,\vec{e}_y - 2S^z\,\vec{e}_z)> \tag{11}$$

describing the averaged motion of particles. If we take adiabatic initial conditions for the spin polarization such that $S_0^x \neq 0$ and $S_0^y = S_0^z = 0$, then for $\vec{r} \ll 1$, the right–hand side of Eq. (10) reduces to the simple harmonic force $\frac{1}{2}\omega_1^2 S_0^x(x\,\vec{e}_x + y\,\vec{e}_y + 8z\,\vec{e}_z)$. Hence Eq. (11) describes the adiabatic motion of particles in a harmonic potential. For $S_0^x < 0$, this is the standard harmonic oscillations of trapped particles, while for $S_0^x > 0$, the latter are not confined and escape from the trap in all directions. Neither of these known cases is of interest for us. Our aim is to find a principally different regime of motion, when the particles are neither completely trapped nor uniformly coupled out but move preferably in one direction forming a collimated beam. Assume that at $t = 0$ the particles are prepared in the state with an initial spin polarization

$$S_0^x = S_0^y = 0, \qquad S_0^z = -S , \tag{12}$$

which is referred to nonadiabatic initial conditions [18]. The ways of preparing such polarized initial states are discussed in detail in Ref. [18]. To take into account the finite size of a device, we introduce the device form–factor, $\varphi(\vec{r})$, and use the notation

$$f(\vec{r}) = \frac{\varphi(\vec{r})}{[(1+2x+x^2+y^2+4z^2)(1+x^2+y^2+4z^2)]^{1/2}}. \tag{13}$$

Then the motion of particles is described by the equation

$$\frac{d^2\vec{r}}{dt^2} = S\omega_1^2 fz\left(x\,\vec{e}_x + y\,\vec{e}_y + 8z\,\vec{e}_z\right). \tag{14}$$

From here it is seen that the motion along the x and y axes is similar to each other. One may also notice that Eq. (14) is invariant under the inversion $z \to -z$ and $S \to -S$. Thence, the trajectories for $S > 0$ are mirror–symmetric, with respect to the $x-y$ plane, to the trajectories for $S < 0$. Therefore, it is sufficient to study only a case, say, when $S > 0$, which will be assumed in what follows. It is also tempting to simplify the equations by passing to dimensionless time measured in units of $(\sqrt{S}\omega_1)^{-1}$. For this purpose, we define $\tau = \sqrt{S}\omega_1 t$. Thus, from (14), we come to the system of equations

$$\frac{d^2x}{d\tau^2} = fxz, \qquad \frac{d^2y}{d\tau^2} = fyz, \qquad \frac{d^2z}{d\tau^2} = 8fz^2. \tag{15}$$

An analytic solution of Eq. (15) is possible only for the initial stage of the motion, when $|\vec{r}| \ll 1$. The corresponding solutions are given by the Weierstrass and Lamé functions [19,20]. From the properties of these

functions, it follows that the axial motion of particles is bounded from below by the minimal value $z_{min} = (z_0^3 - \zeta)^{1/3}$, with $\zeta \equiv (3/16)\dot{z}_0^2$, and $z_0 \equiv z(0)$ being an initial axial position. So, particles can escape only in the positive z–direction. The motion along the axial direction is much faster than along the radial direction. This means that beam collimation begins already at the initial stage of the process. To analyze accurately the whole motion, we have to resort to numerical calculations. Let us specify the form–factor $\varphi(\vec{r})$ assuming a spherical device with $\varphi(\vec{r}) = \exp(-|\vec{r}|^2/L^2)$, in which L is a characteristic size of the device, e.g., the radius of a coil forming magnetic fields. The maximal velocity in the axial direction is $w_{max} \cong (\dot{z}_0 + 2\sqrt{\pi}L)^{1/2}$. Therefore, taking L sufficiently large, it is feasible to get arbitrary strong acceleration in the axial direction. At the same time, the maximal velocity in the radial direction, in the case of $L \gg 1$, is much less than w_{max}, which can be accepted as the definition of collimation.

The results of numerical calculations are presented in characteristic figures corresponding to the initial position at the center of the device and to initial velocities varying in the interval $[-1, 1]$ in all directions. All figures for different $L \gg 1$ are qualitatively the same, because of which we fix here $L = 1000$. Fig. 1 shows the trajectories of particles in the $x - z$ plane at the beginning of motion. It is clearly seen how the particles having initially negative velocities in the axial direction, after reaching the minimal value z_{min}, turn to the positive z direction. In Fig. 2 the trajectories are shown for longer times, demonstrating how a well–collimated narrow beam is formed, being stretched in the axial direction *more than an order of magnitude stronger* than in the radial one. In Fig. 3 the velocities $v(\tau) \equiv \dot{x}(\tau)$ and $w(\tau) \equiv \dot{z}(\tau)$ are pictured, illustrating the acceleration process in the axial direction. When varying L, all figures remain qualitatively the same. The sole thing that changes is the scale on the z–axis. Increasing L by an order squeezes the z scale approximately twice in the figures with trajectories and three times in the figures with velocities. Thus, the maximal velocity in the axial direction in the dimensionless units is $w_{max} = 6$ for $L = 10$; $w_{max} = 19$ for $L = 100$; and $w_{max} = 60$ for $L = 1000$, in accordance with the law $w_{max} \cong (2\sqrt{\pi}L)^{1/2}$.

We would like to recall that equations (15) are derived from (7) which is identical to the initial Eqs. (1) and (2). In the derivation of (15) from (7) the sole assumption used is the existence of the small parameters $|\omega_1/\omega_2| \ll 1$ and $|\omega/\omega_2| \ll 1$. This made it possible to apply the scale separation approach [10-13,16,17] whose mathematical foundation is based on the Krylov–Bogolubov averaging method [14,15].

In order to understand how to choose the characteristic parameters B_1' and B_2 of the magnetic field (3), it is necessary to return to dimensional units. For simplicity, we set $S \sim 1$. Then, the maximal z velocity is $w_{max} \cong (2\sqrt{\pi}L\mu_0 B_2/m)^{1/2}$. To achieve an effective acceleration for a given sort of particles with fixed μ_0 and m, we should take $L \gg 1$ and sufficiently large B_2. However, increasing L and B_2 implies the increase of the size of a device $l = LB_2/B_1'$. Hence, to achieve $L \gg 1$ for a given device requires that $B_2 \ll lB_1'$. At the same time, the existence of the small parameter $|\omega_1/\omega_2| \ll 1$ yields $\hbar^2(B_1')^2/m\mu_0 B_2^3 \ll 1$. Therefore, for a given sort of particles and a given device we must have $(\hbar^2/m\mu_0)^{1/3}(B_1')^{2/3} \ll B_2 \ll lB_1'$. This fixes the required relation between the parameters of the magnetic fields. The frequency of the rotating field, according to Eq. (5), is to be much smaller than ω_2.

The effect described in this paper is rather general, and choosing the corresponding parameters, one could realize such a directed acceleration for any kind of neutral particles having spins. To show that the parameters to be chosen are quite realistic, let us make numerical estimates for ^{87}Rb in a magnetic trap of Ref. [9]. Then, the gradient of the quadrupole field is $B_1' = 120\ G/cm$ and the amplitude of the rotating field is $B_2 = 10\ G$, with the rotating frequency $\omega = 5 \times 10^4 s^{-1}$. This gives $\omega_1 \sim 10^2 s^{-1}$ and $\omega_2 \sim 5 \times 10^7 s^{-1}$. Hence, the inequalities $\omega_1 \ll \omega \ll \omega_2$ hold true. The required small parameters $\omega_1/\omega_2 \sim 10^{-6}$ and $\omega/\omega_2 \sim 10^{-3}$ are really very small, because of which the scale separation approach provides very accurate solutions differing from the exact ones by negligible corrections. The radius of the atom cloud is $R_0 \sim 0.1\ cm$. Taking $L = 10$, the characteristic radius of coils would be $l \sim 1\ cm$. For $L = 100$, this would be $l \sim 10\ cm$. The maximal velocity $w_{max} \approx 60\ cm/s$ for $L = 10$ and $w_{max} \approx 200\ cm/s$ for $L = 100$. Starting from an isotropic distribution of velocities, $\dot{x}_0 \sim \dot{z}_0$, as a result of the preferable acceleration along the z axis, one can obtain the radially squeezed velocity diagram with a rather large squeezing factor. As the phase portraits in Figs. 2 and 3 show, the squeezing factor is about 20 for velocities and 50 for the real phase variables. Let us stress that such a high degree of squeezing is achieved under a rather unfavorable assumption of a spherical device with a spherically symmetric form–factor $\varphi(\vec{r})$. Taking a cigar–shaped device would strongly enhance the degree of collimation. It is also possible to achieve additional squeezing of the beam taking a quadrupole field in Eq. (3) with essentially different field gradients along the axial and radial directions. Thus, one can reach quite high degree of collimation with a squeezing factor of 100 or 1000 and more. The realization of the effect we described is quite feasible by using existing magnetic traps. What one needs to do is to prepare particles in the initial state with polarization (12). This could be achieved in several ways. For example, one could prepare particles in the desired state inside one trap and then quickly load them into another trap with the considered magnetic field [18]. Or it might be possible to form the necessary initial polarization by means of a short pulse.

The presented calculations are based on single–particle trajectories. Many–particle physics, related to collisions of particles, has not been treated. Since we would like to stress the practical applicability of the considered mechanism, it is useful to give at least a coarse estimate on what happens if collisions are included. And the most important is to

define conditions when the collisions of particles do not essentially disturb the single–particle picture and permit one to realize the semiconfining regime of motion, when particles move predominantly in one direction.

In order to include particle collisions, we need to add to the right–hand side of the evolution equation (14) an additional term describing an effective force caused by these collisions. It is customary to treat the collisional force as a random variable. To this end, we can model this force by a random vector $\gamma \vec{\xi}$, where γ is a collision rate and $\vec{\xi} = \{\xi_x, \xi_y, \xi_z\}$ is a stochastic vector variable. Following again the common way, we may interpret the set $\{\xi_\mu(t)\}$, with $\mu = x, y, z$, as a set of Gaussian random variables characterized by the stochastic averages

$$\ll \xi_\mu \gg = 0 \; , \qquad \ll \xi_\mu(t)\xi_\nu(t') \gg = 2D_\mu \delta_{\mu\nu}\delta(t-t') \; , \tag{16}$$

in which D_μ is a diffusion rate in the μ direction. Adding the random collisional force to the right–hand side of the evolution equation (14), we have, instead of Eq. (15), the system of equations

$$\frac{d^2x}{dt^2} = S\omega_1^2 fzx + \gamma\xi_x \; ,$$

$$\frac{d^2y}{dt^2} = S\omega_1^2 fzy + \gamma\xi_y \; , \tag{17}$$

$$\frac{d^2z}{dt^2} = 8S\omega_1^2 fz^2 + \gamma\xi_z \; .$$

As is evident, if particle collisions are intensive, so that the motion is dominated by the random collision terms, then no organized motion of particles coherently moving in one direction is possible. The semiconfining regime can survive only if influence of collisions is sufficiently weak, so that the random terms in Eqs. (17) can be treated as perturbation. In such a case, the solutions to Eqs. (17) can be presented as

$$x = x_1 + x_2 \; , \qquad y = y_1 + y_2 \; , \qquad z = z_1 + z_2 \; ,$$

where x_1, y_1, and z_1 are the solutions of the unperturbed Eqs. (15) and x_2, y_2, and z_2 are the solutions to the equations

$$\frac{d^2x_2}{dt^2} = S\omega_1^2 f(z_1x_2 + x_1z_2) + \gamma\xi_x \; ,$$

$$\frac{d^2y_2}{dt^2} = S\omega_1^2 f(z_1y_2 + y_1z_2) + \gamma\xi_y \; , \tag{18}$$

$$\frac{d^2z_2}{dt^2} = 16S\omega_1^2 fz_1z_2 + \gamma\xi_z \; .$$

The unperturbed functions x_1, y_1, and z_1 can be considered as slow compared to the random functions x_2, y_2, and z_2. Keeping in Eqs. (18) the unperturbed functions as quasi–invariants, one obtains the solutions

$$x_2(t) = \int_0^t G_x(t-t')\left[\gamma\xi_x(t') + S\omega_1^2 fx_1z_2(t')\right]\, dt' \; ,$$

$$z_2(t) = \int_0^t G_z(t-t')\gamma\xi_z(t')\, dt' \; ,$$

where the solution $y_2(t)$, having the form similar to $x_2(t)$, is not written down and

$$G_x(t) = \frac{\sinh(\omega_x t)}{\omega_x} \; , \qquad G_z(t) = \frac{\sinh(\omega_z t)}{\omega_z} \; , \qquad \omega_x = \sqrt{S\omega_1^2 fz_1} \; , \qquad \omega_z = 4\omega_x \; .$$

According to the stochastic averages (16), we have

$$\ll x_2(t) \gg = \ll z_2(t) \gg = 0 \; ,$$

$$\ll x_2^2(t) \gg = \frac{\gamma^2 D_x t}{\omega_x^2}\left[\frac{\sinh(2\omega_x t)}{2\omega_x t} - 1\right] +$$

$$+ \frac{\omega_x^4 x_1^2 \gamma^2 D_z t}{\omega_z^2(\omega_z^2-\omega_x^2)^2 z_1^2}\left\{\cosh(\omega_x t)\cosh(\omega_z t) + \frac{\sinh(\omega_z t)}{\omega_z t}\left[\cosh(\omega_z t) - \cosh(\omega_x t)\right] - \right.$$

$$\left. - \frac{\omega_z}{\omega_x}\sinh(\omega_x t)\sinh(\omega_z t) - 1\right\} , \tag{19}$$

$$\ll z_2^2 \gg = \frac{\gamma^2 D_z t}{\omega_z^2}\left[\frac{\sinh(2\omega_z t)}{2\omega_z t} - 1\right] .$$

As is seen from here, the collisions will not disturb much the ordered motion of particles provided that

$$\frac{\gamma^2 D}{\omega_1^3} \ll 1 , \qquad D \equiv \sup\{D_x,\ D_y,\ D_z\} .$$

If we take for estimates the collision rate as $\gamma \sim \hbar\rho a_0/m$, where ρ is the density of particles and a_0 is a scattering length, and the diffusion rate as $D \sim k_B T/\hbar$, where T is temperature, then we get the inequality

$$\frac{\hbar\rho^2 a_0^2 k_B T}{m^2\omega_1^3} \ll 1 . \tag{20}$$

The latter shows that the influence of random particle collisions, disturbing the organized semiconfined motion, can be negligible if density, temperature, or the scattering length are small enough to satisfy condition (20). When inequality (20) does not hold, the organized directed motion of particles will be essentially spoiled by collisions. Then the motion becomes more complicated, at the same time becoming of no interest for our purpose. Our aim here has been to find the conditions when the directed semiconfined motion of particles is possible. Such a regime looks like feasible since one always can satisfy condition (20) by varying the parameters of the system.

Concluding, we have advanced a novel general mechanism for creating *well–collimated* beams of neutral particles by means of *magnetic fields*. Such particles could be neutrons or neutral atoms and molecules with nonzero spin. In particular, such a mechanism can be employed for creating narrow beams of molecules in molecular–beam masers or directed beams of neutral atoms for other purposes. The mechanism does not depend on statistics and can be used for Bose as well as for Fermi particles. Varying the magnetic field parameters and the shape of a device, one can regulate the beam characteristics in wide limits, achieving the desired degree of collimation.

---

[1] A. Septier, ed., Focusing of Charged Particles (Academic, New York, 1967).
[2] C.H. Townes, ed., Quantum Electronics (Columbia Univ., New York, 1960).
[3] M. Brotherton, Masers and Lasers (McGraw–Hill, New York, 1964).
[4] M. Sargent, M.O. Scully, W.E. Lamb, Laser Physics (Addison–Wesley, London, 1977).
[5] V.I. Yukalov, E.P. Yukalova, Bull. Am. Phys. Soc. 43 (1998) 237.
[6] L.D. Landau, E.M. Lifshitz, Quantum Mechanics (Pergamon, Oxford, 1987).
[7] D.E. Pritchard, Phys. Rev. Lett. 51 (1983) 1336.
[8] H.F. Hess, Phys. Rev. 34 (1986) 3476.
[9] W. Petrich, M.H. Anderson, J.R. Ensher, E.A. Cornell, Phys. Rev. Lett. 74 (1995) 3352.
[10] V.I. Yukalov, Laser Phys. 3 (1993) 870.
[11] V.I. Yukalov, Laser Phys. 5 (1995) 970.
[12] V.I. Yukalov, Phys. Rev. Lett. 75 (1995) 3000.
[13] V.I. Yukalov, Phys. Rev. B 53 (1996) 9232.
[14] N.M. Krylov, N.N. Bogolubov, Introduction to Nonlinear Mechanics (Princeton Univ., Princeton, 1947).
[15] N.N. Bogolubov, Y.A. Mitropolsky, Asymptotic Methods in the Theory of Nonlinear Oscillations (Gordon and Breach, New York, 1962).
[16] V.I. Yukalov, Phys. Rev. A 56 (1997) 5004.

[17] V.I. Yukalov, Laser Phys. 7 (1997) 998.
[18] C. Cohen–Tannoudji, B. Diu, F. Laloë, Quantum Mechanics (Wiley, New York, 1977).
[19] E.T. Whittaker, G.N. Watson, Modern Analysis (Cambridge Univ., Cambridge, 1988).
[20] H. Bateman, A. Erdélyi, Higher Transcendental Functions (Krieger, Malabar, 1987), Vol.3.

Figure Captions

Fig. 1. The trajectories of particles at the initial stage of the acceleration process, for $0 \leq \tau \leq 5$.

Fig. 2. The trajectories of particles for $0 \leq \tau \leq 20$.

Fig. 3. The velocities of particles in the radial, $v(\tau)$, and axial, $w(\tau)$, directions for $0 \leq \tau \leq 100$.